Roots of Polynomials,
Sketch for Presentation

- State the problem: is there a "nice" formula which would produce the roots of a polynomial of degree n?
- Explain what "nice" means: by "nice" we mean "in terms of radicals", that is only the operations of addition, subtraction, multiplication, division and taking the n-th root can be used in the formula.
- State the answer: yes, if the polynomial is of degree 4 or less.
- State the formulas:

  ♣ $ax + b$, $x = -b/a$.

  ♢ $ax^2 + bx + c$, $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

  ♡ In degree 3 we consider only polynomials of the form $x^3 + ax^2 + bx + c$ (explain why!). We can use the substitution $x = x - \frac{a}{3}$ to reduce it to the form $x^3 + px + q$, where $p = b - \frac{a^2}{3}$ and $q = \frac{2}{27}a^3 - \frac{ab}{3} + c$. Now the formula is

  $$x_{1,2,3} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}.$$

  ♠ $ax^4 + bx^3 + cx^2 + dx + e$ can be reduced to $x^3 - 2px^2 + (p^2 - 4r)x + q^2$ by a rather messy substitution. Now we use the formula for degree 3 polynomials to find three roots x_1, x_2 and x_3. Four of the eight numbers

  $$\frac{\pm\sqrt{x_1} \pm \sqrt{x_2} \pm \sqrt{x_3}}{2}$$

  will be the roots of the degree 4 polynomial we started with.

- (If there is time) What about polynomials of degree 5 or larger? There is no "nice" formula. This is a classic result of a branch of mathematics called "Galois Theory". This theory studies what happens to the set of rational numbers once we adjoin to it one or several (or infinitely many!) irrational numbers which are roots of some polynomials with rational coefficients.

  An example of an irrational number which is a root of a polynomial with rational coefficients is $\sqrt{2}$. It is a root of $x^2 - 2$. An example of an irrational number which is NOT a root of any polynomial with rational coefficients is $e = 2.71828\ldots$.