Infinity and beyond

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1 Introduction

The concept of infinity has been around probably since people started to count things. While learning to count, almost everyone has experienced the striking realisation that if there was no need for food and sleep, one could keep counting forever.

The Greek and early Indian cultures were the first to make some progress with understanding what infinity is. However, not until the second half of the 19th century have there been any significant breakthroughs. The bulk of our understanding of infinity was generated by the work of Georg Cantor, a German mathematician. He was among the first to notice that there are actually many different kinds of infinity. Cantor's work was championed by the most influential mathematician of the late 19th and early 20th century, David Hilbert.

Working with infinity is notoriously difficult because human intuition is not equipped to deal with infinite quantities. One classic thought-experiment that demonstrates the failure of human intuition when dealing with infinity is called "Hilbert's hotel" and it describes the following situation: somewhere in space there is a hotel with an infinite number of rooms and all the rooms are always booked. A new guest approaches the hotel and asks the host if she could accommodate her. I'm sorry, the host says, but we are all booked. No problem, the guest, a brilliant mathematician, says. Simply tell the guest in the first room to move to the second room and to tell the guest in the second room to move to the third room and so on. All your guests will still be taken care of and the first room will become available.

Other classic problems are the following: are there more positive integers than positive and even integers? Are there more fractions of integers than integers? Both of these problems (and more) will be answered in this project.

In this project we will demonstrate the difference between three "types" of sets: finite, countably infinite and uncountably infinite. Only basic familiarity with functions is required.

2 Sets

Sets are collections of objects. They are usually denoted by capital or blackborad-bold letters, and the objects belonging to a set are enclosed by curly brackets. For example, $A = \{1, 3, 5\}$ is a set consisting of the integers 1, 3, and 5. We also say that 1 is an element of A and denote by $1 \in A$. The order and the multiplicity of the elements in a set is not relevant. Therefore, $\{1, 1, 5, 3, 3, 3\}$ is the same set as $\{1, 3, 5\}$.

We will consider the following sets of numbers:

$\mathbb{N}$, the set of natural numbers, i.e. positive integers,

$\mathbb{Z}$, the set of integers,

$\mathbb{Q}$, the set of rational numbers, i.e. the set of fractions of integers

$\mathbb{R}$, the set of real numbers,

$\mathbb{C}$, the set of complex numbers, i.e. the set of all numbers $a + bi$, where a and b are real numbers and i denotes the imaginary number $\sqrt{-1}$.

We will also consider subsets of these sets, that is the sets which consist of some (possibly infinitely many) elements of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. For example, $\{-1, 3, 5\}$ is a subset of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ but not of $\mathbb{N}$.

3 Finite sets

It is intuitively clear when a set is finite; precisely when it has a finite number of elements. However, we would prefer to express the statement "has a finite number of elements" in a more mathematical way. For that purpose we introduce the notion of a bijective function.

3.1 Injections, surjections and bijections

We will denote functions with expressions of the type $f\colon A \to B$. This means that f maps elements of the set A into the elements of the set B. For example, $f\colon \{1, 2, 3\} \to \{-1, -2, -3\}$ is a function that maps elements of the set $\{1, 2, 3\}$ into the elements of the set $\{-1, -2, -3\}$. To properly define a function we should also specify a mapping rule; for example, $f(k) = -k$, where k is any element of $\{1, 2, 3\}$. If a is an element of A, then the value $f(a)$ of the function f at a is called the *image* of a.

An *injective*, or a 1-1 (one-one) function is a function $f\colon A \to B$ which satisfies the following property:

$$\text{for all } a, b \in A \text{ with } a \neq b, \text{ we have } f(a) \neq f(b).$$

In other words, an injective function is precisely the function which for distinct inputs gives distinct outputs.

Problem 1. Is the function $f\colon \{1, 2, 3\} \to \{-1, -2, -3\}$ with the mapping rule $f(k) = -k$ an injection?

Problem 2. Is the function $f\colon \mathbb{N} \to \{0\}$ with the mapping rule $f(n) = 0$ an injection?

A *surjective*, or an onto function is a function $f\colon A \to B$ which satisfies the following property:

$$\text{for all } b \in B \text{ there exists } a \in A \text{ such that } f(a) = b.$$

In other words, a surjective function is precisely the function for which every element of B is the image of some element in A.

Problem 3. Is the function $f\colon \{1,2,3\} \to \{-1,-2,-3\}$ with the mapping rule $f(k) = -k$ a surjection?

Problem 4. Is the function $f\colon \mathbb{N} \to \{0\}$ with the mapping rule $f(n) = 0$ a surjection?

Problem 5. Is the function $f\colon \mathbb{N} \to \{0,1,2\}$ with the mapping rule $f(n) = 0$ a surjection?

A *bijective* function, also called a 1-1 correspondence, is a function which is both injective and surjective. That is, it is a function $f\colon A \to B$ with the property that distinct inputs give distinct outputs and that every element in B is the image of some element in A.

Problem 6. Is the function $f\colon \{1,2,3\} \to \{-1,-2,-3\}$ with the mapping rule $f(k) = -k$ a bijection?

Problem 7. Is the function $f\colon \mathbb{N} \to \mathbb{N}$ with the mapping rule $f(n) = n$ a bijection?

Problem 8. Is the function $f\colon \mathbb{Z} \to \mathbb{Z}$ with the mapping rule $f(n) = -n$ a bijection?

3.2 Size of a finite set

We say that a set A is *finite* if there is a bijection $f\colon \{1,\dots,n\} \to A$, where n is some positive integer. If such a bijection exists, we also say that the *size* or *cardinality* of A is n, denoted by $|A| = n$. There is only one set of size 0, namely the *empty set* $\emptyset = \{\}$.

It may seem that this definition is far removed from our intuitive understanding of finiteness, however everything becomes clear if we think of bijections in the following way: suppose we are given a set $A = \{a_1, a_2, \dots, a_j\}$. We don't know the value of j. Now let n be some positive integer and let us draw the following table:

1	2	...	n

Now let's take some element of A, for example a_1, and put it into some box in the bottom row of the table. Just to make things more interesting, let's put it in the second box in the bottom row.

1	2	...	n
	a_1		

Now let's do the same thing with some other element of A, for example a_7. Let's place it into the first box in the bottom row:

1	2	...	n
a_7	a_1		

We repeat this process until we either exhaust all the boxes in the bottom row or we exhaust all the elements of A. There are three possibilities that can occur:

- We've exhausted all the elements of A and there are still some boxes left empty, let's say r of them.

1	2	...	$n-r-1$	$n-r$	...	n
a_7	a_1	...	a_j			

  In this case, delete the last r columns of the table to get the table of the form

1	2	...	$n-r$
a_7	a_1	...	a_j

  This means that we can define a bijection $f\colon \{1, 2, \ldots, n-r\} \to A$ where the mapping rule is given simply by mapping the numbers in the top row to the elements of A in the bottom row.

- We've exhausted all the elements of A and there are no boxes left.

1	2	...	n
a_7	a_1	...	a_j

  As in the last case, we can define a bijection $f\colon \{1, 2, \ldots, n\} \to A$ where the mapping rule is given simply by mapping the numbers in the top row to the elements of A in the bottom row.

- There are no empty boxes left but we haven't exhausted all the elements of A. In that case we need a larger n. If no such n ca be found, then we say that A is *not finite.*

4 Infinite sets

The last case of the previous section hinted at a definition of an infinite set: a set A is infinite if it is not finite, that is if there is no bijection $f\colon \{1,2,\dots,n\} \to A$ for all positive integers n. However, we would like a positive definition where the word "not" is not mentioned.

We say that a set A is *infinite* if there is a bijection $f\colon \mathbb{N} \to A$. In that case, we say that the size or cardinality of A is *aleph-null* and denote it by $\aleph_0$.

Clearly, the set $\mathbb{N}$ is itself infinite by problem 7. What about the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$? Since $\mathbb{N}$ is a subset of all the aforementioned sets, it should be obvious that all of those sets are infinite. However, the situation is much more delicate than it seems.

Problem 9. Let $2\mathbb{N} = \{2,4,6,\dots\}$ be the set of all positive even integers. Show that there are as many positive even integers as there are positive integers. Modify the argument to show that there are as many positive odd integers as there are positive integers.

Problem 10. Show that there is a bijection $f\colon \mathbb{N} \to \mathbb{Z}$.

By this point it should not be surprising that various infinite sets consisting only of integers are all equal in size. What is not entirely obvious is that there are as many rational numbers as there are integers. To see this, arrange the set $\mathbb{Q}_+$ of positive rational numbers in the following way:

1	1/2	1/3	1/4	...
2	2/2	2/3	2/4	...
3	3/2	3/3	3/4	...
⋮	⋮	⋮	⋮	

We will transform this table into a single line in the following way: start at 1 and move down to 2. Now go up-right to 1/2. For now we have the line 1, 2, 1/2. Move right from 1/2 to 1/3. Now move down-left until you reach the left wall. You have to pass through 2/2 and 3. So now we have the line 1, 2, 1/2, 1/3, 2/2, 3. We repeat this process by starting at 3, moving down to 4 and then go up-right until we reach the top wall. Once we reach the top wall, we move to the next right number and then go down-left until we reach the left wall. It should now be clear that in this way we can list all the positive rational numbers as the sequence

$$1, 2, 1/2, 1/3, 2/3, 3, 4, 3/4, \ldots$$

To form a bijection $f\colon N \to \mathbb{Q}_+$ we simply map 1 to 1, 2 to 2, 3 to 1/2 etc. However, we are not finished yet. We have only shown that there are as many positive rational numbers as there are positive integers. We have seen in problem 10 that there is a bijection $g\colon \mathbb{N} \to \mathbb{Z}$ so if we form a bijection $h\colon \mathbb{Z} \to \mathbb{Q}$, we will have a bijection $h \circ g\colon \mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$.

So far, we have a bijection f that maps the positive integers to positive rational numbers. We will extend this bijection to the function f_0 so that f_0 maps 0 to 0 and behaves as f on all positive integers. Now let h be a function that behaves as f_0 on the set $\{0, 1, 2, \ldots\}$ and as $-f$ on the set $\{-1, -2, -3, \ldots\}$, that is, h maps -1 to -1, -2 to -2, -3 to $-1/2$ etc. Therefore, h bijectively maps positive integers to positive rational numbers, negative integers to negative rational numbers and 0 to 0 so it is a bijection $h\colon \mathbb{Z} \to \mathbb{Q}$. Hence, there are as many rational numbers as there are integers, and since there are as many integers as positive integers, there are as many rational numbers as there are positive integers.

4.1 Uncountable infinity

So far we have seen that various sets of integers and fractions of integers are equal in size. One might hope that the set $\mathbb{R}$ is also of the same size. However, that is not the case. How do we see that? We will show that there are more numbers in the interval $0 < x < 1$ then there are positive integers.

Suppose that it is the case that there are as many numbers in the interval $0 < x < 1$ as there are positive integers. That means that there is a bijection $f\colon \mathbb{N} \to \langle 0, 1\rangle$. Let's form the following table:

$f(1) =$	$0.a_{11}a_{12}a_{13}\ldots$
$f(2) =$	$0.a_{21}a_{22}a_{23}\ldots$
$f(3) =$	$0.a_{31}a_{32}a_{33}\ldots$
$\vdots$	$\vdots$

Now let us choose numbers b_1, b_2, b_3, $\ldots$ such that none of them is 9 and $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$, etc. Let x be the number $x = 0.b_1b_2b_3\ldots$. It is clearly in the interval $0 < x < 1$ (note that by requiring that none of the b_i

are equal to 9 we avoided the case that $x = 0.999\cdots = 1$) and not equal to any of the $f(1)$, $f(2)$, $f(3)$ etc. since it differs from $f(i)$ in the digit $b_i \neq a_{ii}$. Therefore, the function f cannot be a surjection since there is no element in $\mathbb{N}$ that would map to x. Hence it cannot be a bijection.

This means that there is no bijection $f\colon \mathbb{N} \to \langle 0, 1\rangle$ and since the interval $0 < x < 1$ is contained in $\mathbb{R}$, there is no bijection $f\colon \mathbb{N} \to \mathbb{R}$. So either $\mathbb{R}$ is finite, which is ridiculous since $\mathbb{N}$ is a subset of $\mathbb{R}$, or it is larger in size than $\mathbb{N}$. We say that $\mathbb{R}$ is *uncountably infinite* and that its size is beth-one, denoted by $\beth_1$.

If A is a set and there is a bijection $f\colon \mathbb{R} \to \mathbb{A}$, then we say that A is uncountably infinite.

Problem 11. Show that the set of all non-negative real numbers is uncountably infinite.

In contrast to uncountable infinity, we say that sets of the size $\aleph_0$, that is the sets of (positive) integers, rationals etc. are countably infinite. The name suggests that we can "count" the elements of those sets, that is, given an element of a countably infinite set, we know its immediate successor while if we are given an element of an uncountable infinite set, we do not know its immediate successor.

4.2 Size of the set of complex numbers

So far we have found that the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are equal in size and that $\mathbb{R}$ is larger in size. What about the complex numbers? The following problem hints that there are as many complex numbers as real numbers:

Problem 12. Show that the function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ given by the mapping rule $f(x, y) = x + iy$ is a bijection.

However, we do not know that $\mathbb{R} \times \mathbb{R}$ is uncountably infinite. If we could show that $\mathbb{R} \times \mathbb{R}$ is uncountably infinite then we would have a bijection $g\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. Since by problem 12 we have a bijection $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, we would have a bijection $f \circ g\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and we could conclude that the set $\mathbb{C}$ of complex numbers is indeed uncountable.

Unfortunately, it is not so easy to construct a bijection $g\colon \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. Such bijections do indeed exist and one of the most visual ones are called space-filling curves. One space filling curve is illustrated in figure 1. Therefore, there are as many complex numbers as there are real numbers.

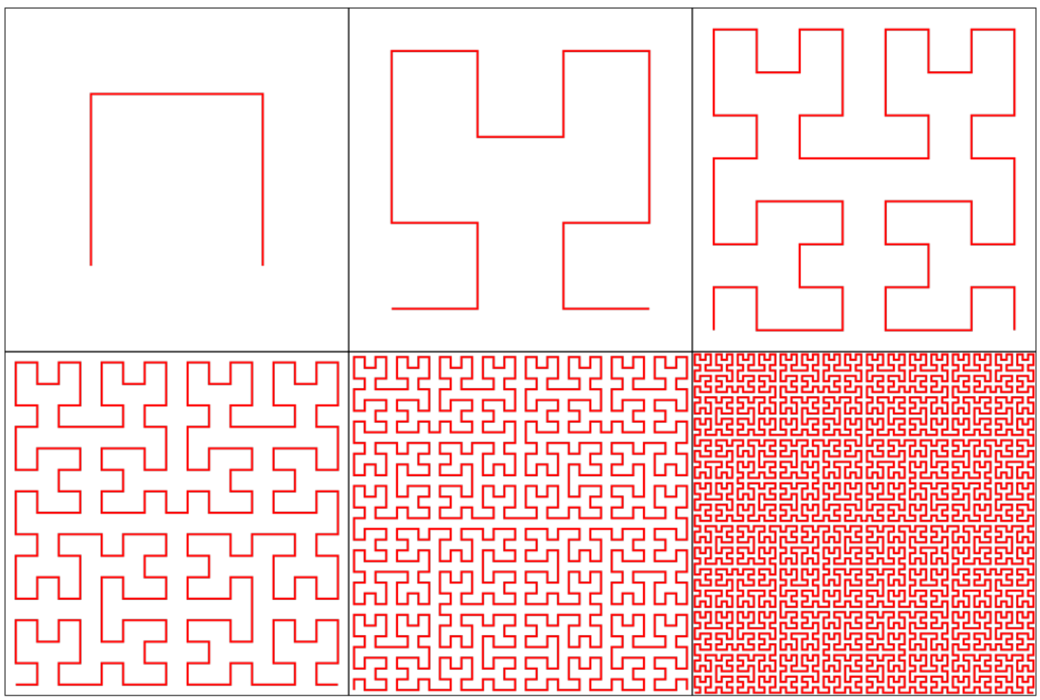

Figure 1: A space-filling curve

5 Epilogue

The study of infinity has produced some of the most spectacular results in mathematics. One of the most inspiring problems in mathematics is the so-called *Continuum hypothesis* which asks if there is a set which is bigger in size than $\mathbb{N}$ but smaller than $\mathbb{R}$. It was posed by Cantor in 1878 and is still not resolved to the satisfaction of all. Today it is known that, within the standard mathematical framework, the Continuum hypothesis cannot be proven or disproven. Although this answer doesn't say anything about the hypothesis itself it still remains a very powerful result as it shows that there are limits to what we can learn using mathematics.